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Dirac and Infeld-Hull ladder operator methods for a modified oscillator potential

O Babynanda Devi* and C Amuba Singh

Department of Physics, Manipur University, Canchipur, Imphal-795 003, Manipur, India

E-mail amuchab@rediffmail.com

Abstract Dirac's method of solving the Schrodinger eigenvalue problem for the harmonic oscillator by constructing eigenvalue raising and lowering operators for a potential of the form $V(x) = Ax^2 + Bx^{-2}$ is reviewed The eigenvalue problem of the potential is also solved by applying the Infeld and Hull's factorization method In the process we construct the Infeld- Hull ladder operators which connect eigenvectors of a hierarchy of Hamiltonians The laddering actions in the Dirac method and Infeld – Hull method are compared and constrasted

Keywords : Dirac ladder operators, Infeld-Hull ladder operators, Infeld-Hull factorization

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1. Introduction

In the early days of quantum mechanics Dirac[1] developed the elegant operator method of finding the energy eigenvalues and eigenvectors of the harmonic oscillator by constructing the eigenvalue raising and lowering operators. This celebrated method could be found in most text books [2] on nonrelativistic quantum mechanics. Later, generalizing Schrodinger's approach of 'solving eigenvalue problems by factorization' [3], Infeld and Hull [4] developed an operator factorization method which enables us to solve the eigenvalue problem of a large class of second order differential equations algebraically. In this method the original differential operator is a function of a parameter m and the factorization essentially provides a laddering process which takes a solution for certain value of m to solutions for $m \pm 1$.

Even as the factorization method of Infeld and Hull is very versatile and, in fact, lies at the very foundation for the latter day development of the Supersymmetric Quantum Mechanics [5], as for the linear harmonic oscillator system the Dirac approach is more

Corresponding Author

straightforward and directly generates the higher eigenfunctions from the ground-state eigenfunction. Moreover in the Dirac method the normalization constants are easily determined without having to evaluate cumbersome integrals.

In [6] we have extended Dirac's method to find the energy eigenvalues and normalized eigenfunctions for the potential $V(x) = Ax^2 + Bx^{-2}$. Section 2 gives a brief review of this work.. In Section 3 we apply the Infeld- Hull factorization method to the above potential In Section 4 we factorise the Hamiltonian in four different ways thereby identifying four different Infeld-Hull ladder operators and their laddering actions is explained. A detailed comparison of the laddering actions in the Infeld – Hull method and the Dirac method is given. The comparison also explains why the Dirac ladder operator for the system is necessarily quadratic in the momentum.

The present work also carries an Appendix A where we discuss the revelation of unphysical spectrum when the requirement of normalisability is relaxed.

2. Dirac ladder operator method for the potential $V(x) = Ax^2 + Bx^{-2}$

For a Hamiltonian H Dirac ladder operators $Q_\pm$ are linear operators, which satisfy the commutation relations

$$[H, Q_\pm] = \pm\varepsilon\, Q_\pm \tag{1}$$

where ε is a real constant having dimension of energy and signifies the quantum of energy by which the ladder operator would shift the energy eigenvalue. The basic laddering property of $Q_\pm$ is that – if $H\Psi_E = E\Psi_E$, then $H(Q_\pm \Psi_E) = (E \pm \varepsilon)(Q_\pm \Psi_E)$, provided $Q_\pm$ does not annihilate Ψ_E. Obviously, Q_+ and Q_- are essentially the hermitian conjugates of each other. Since ε is a parameter independent of the state of the physical system, existence of Dirac ladder operators implies that the Hamiltonian has equi-spaced eigenvalue spectrum. And it is obvious that Dirac ladder operators may not exist for an arbitrary Hamiltonian. Our definition, eq. (1), also rules out state-dependent ladder operators which are constructed adhoc, such as the ones in Refs. [7, 8]. As in [6] we consider a harmonic oscillator subjected to a repulsive force varying as x^{-3}, which is described by the Hamiltonian

$$H = \frac{1}{2}\hbar\omega\left(p^2 + x^2 + \beta(\beta-1)x^{-2}\right), \tag{2}$$

where x is the dimensionless position variable and $p = -id/dx$ and β is a dimensionless parameter ($\beta > 1$). The eigenvalue problem can be algebraically solved [6] by constructing the ladder operators Q and $Q^\dagger$, where

$$Q = \left(H/\hbar\omega - ixp - x^2 - \frac{1}{2}\right) \tag{3}$$

These operators H, Q and $Q^\dagger$ satisfy the commutation relations

$$\left[H, Q^\dagger\right] = 2\,\hbar\omega\, Q^\dagger, \left[H, Q\right] = -2\hbar\omega\, Q, \left[Q, Q^\dagger\right] = 4H/\hbar\omega. \tag{4}$$

That is, $Q^\dagger$ and Q are Dirac ladder operators respectively raising and lowering the eigenvalues of H by $2\hbar\omega$. Factorizing the Hamiltonian as

$$H = \hbar\omega\left(A^\dagger A + \beta + \frac{1}{2}\right), \quad \text{with} \quad A = \frac{1}{\sqrt{2}}(p - ix + i\beta / x) \tag{5}$$

and using the algebra eq.(4) it can be shown [6] that the energy eigenvalues are

$$E_n = \hbar\omega\left(2n + \beta + \frac{1}{2}\right) \tag{6}$$

and that the normalized eigenfunctions are

$$\Psi_n(x) = C_n x^\beta e^{-\frac{1}{2}x^2} L_n^{\beta - \frac{1}{2}}, \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots. \tag{7}$$

where $L_n^\alpha(z)$ is the Laguerre polynomial [9] and $C_n = \left(2n!/\Gamma\left(\beta + \frac{1}{2}\right)\right)^{\frac{1}{2}}$.

The reduced radial Schrödinger equation for a central potential $V(r) = Ar^2 + Br^{-2}$ is identical, in form, to the one dimensional Schrödinger eigenvalue equation for the potential $V(x) = A\,x^2 + Bx^{-2}$. Therefore the energy eigenvalues and *normalized* wavefunctions obtained above can be readily adapted for this central potential without having to solve the Schrödinger equation. An added advantage of this method over the method of direct solution [10,11] of the Schrödinger equation is that the normalization constants of the wavefunctions are automatically determined once we normalize the groundstate wavefunction.

3. Application of the Infeld-Hull factorisation

Treating β as a parameter we rewrite the Hamiltonian of eq. (2) in the factorised form as

$$H(\beta) = \hbar\omega\left(A^\dagger(\beta)\, A(\beta) + \beta + \frac{1}{2}\right), \tag{8}$$

with $$A(\beta) = \frac{1}{\sqrt{2}}\left(p - ix + i\beta\, x^{-1}\right). \tag{9}$$

We then have

$$A(\beta)\, A^\dagger(\beta) \frac{H(\beta + 1)}{\hbar\omega} - \beta + \frac{1}{2}. \tag{10}$$

The eigenvalue equation $H(\beta)\Psi_E = E\Psi_E$ can be written as

$$A^{\dagger}(\beta)\, A(\beta)\Psi_E = \left(\lambda - \beta - \frac{1}{2}\right)\Psi_E, \tag{11}$$

where, $\lambda = E/\hbar\omega$. (12)

Applying $A(\beta)$ on both sides of eq. (11) and using eq. (10) we obtain

$$H(\beta+1)\, A(\beta)\Psi_E = (E - \hbar\omega)\, A(\beta)\Psi_E\,. \tag{13}$$

That is, $A(\beta)$ acting on the eigenvector Ψ_E of $H(\beta)$ produces an eigenvector of $H(\beta+1)$ belonging to the eigenvalue $(E-\hbar\omega)$. Further eq. (11) gives

$$\langle A(\beta)\Psi_E | A(\beta)\Psi_E\rangle = \left(\lambda - \beta - \frac{1}{2}\right)\langle\Psi_E|\Psi_E\rangle\,. \tag{14}$$

Assuming Ψ_E to be normalizable $A(\beta)\,\Psi_E$ is also normalizable provided $\lambda \geq \beta + 1/2$ that is, if $E \geq (\beta + 1/2)\,\hbar\omega$. Introducing the vectors

$$\Phi_{\beta+m+1} = A(\beta+m)\,\Phi_{\beta+m}, \qquad m = 0,\ 1,\ 2,\ 3,\ \ldots\ldots, \tag{15}$$

with $\Phi_\beta \equiv \Psi_E$, it is easy to show that

$$H(\beta+m)\Phi_{\beta+m} = (E - m\,\hbar\omega)\Phi_{\beta+m}\,. \tag{16}$$

Then eq. (14) is rewritten as

$$\langle\Phi_{\beta+1}|\Phi_{\beta+1}\rangle = \left(\lambda - \beta - \frac{1}{2}\right)\langle\Psi_E|\Psi_E\rangle\,. \tag{17}$$

Similarly,

$$\langle\Phi_{\beta+2}|\Phi_{\beta+2}\rangle = \langle\Phi_{\beta+1}|\,\frac{H(\beta+1)}{\hbar\omega} - \beta - 1 - \frac{1}{2}|\Phi_{\beta+1}\rangle$$

$$= \left(\lambda - \beta - \frac{1}{2} - 2\right)\left(\lambda - \beta - \frac{1}{2}\right)\langle\Psi_E|\Psi_E\rangle\,.$$

The process can be formally continued to get

$$\langle\Phi_{\beta+m+1}|\Phi_{\beta+m+1}\rangle = \left(\lambda - \beta - \frac{1}{2} - 2m\right)\left(\lambda - \beta - \frac{1}{2} - 2(m-1)\right)\ldots\left(\lambda - \beta - \frac{1}{2}\right)\langle\Psi_E|\Psi_E\rangle\,. \tag{18}$$

To avoid the possibility of the norm-square becoming negative the process of successive generation of the Φ - vectors has to terminate at some stage, say at $m = n$ and that can happen only if

$$\lambda = \beta + \frac{1}{2} + 2n, \qquad n = 0, 1, 2, 3, \ldots\ldots,$$

that is, if $$E = \hbar\omega\left(\beta + \frac{1}{2} + 2n\right); \tag{19}$$

and in that case $A(\beta + n)\,\Phi_{\beta+n} = 0$.

Eq. (19) gives the eigenvalues of the Hamiltonian $H(\beta)$. Obviously the ground-state Ψ'_0 of $H(\beta)$ is defined by the requirement $A(\beta)\Psi'_0 = 0$.

4. Dirac ladder operatos *vs* Infeld-Hull ladder operators

We have seen that in the Dirac approach (Section 2) the ladder operator $Q^{\dagger}$ takes an eigenstate of $H(\beta)$ to the next higher eigenstate of the same Hamiltonian. In the factorization method of Infeld and Hull the operator $A(\beta)$ takes an eigenstate of $H(\beta)$ to an eigenstate of $H(\beta+1)$.

The Hamiltonian

$$H(\beta) = \hbar\omega\,\frac{1}{2}\left(p^2 + x^2 + \beta(\beta-1)\,x^{-2}\right)$$

can be factorized in four different ways (including the one given in eq. (8)) :

$$H(\beta) = \hbar\omega\left(A^{\dagger}(\beta)\,A(\beta) + \beta + \frac{1}{2}\right), \tag{20a}$$

$$H(\beta) = \hbar\omega\left(B^{\dagger}(\beta)\,B(\beta) - \beta - \frac{1}{2}\right), \tag{20b}$$

$$H(\beta) = \hbar\omega\left(C^{\dagger}(\beta)\,C(\beta) + \beta - \frac{3}{2}\right), \tag{20c}$$

and, $$H(\beta) = \hbar\omega\left(D^{\dagger}(\beta)\,D(\beta) - \beta + \frac{3}{2}\right), \tag{20d}$$

where, $$A(\beta) = \frac{1}{\sqrt{2}}\left(p - ix + i\beta\,x^{-1}\right), \quad B(\beta) = \frac{1}{\sqrt{2}}\left(p + ix + i\beta\,x^{-1}\right), \tag{21}$$

and $$C = \frac{1}{\sqrt{2}}\left(p + ix - i(\beta - 1)\, x^{-1}\right), \text{ and } D = \frac{1}{\sqrt{2}}\left(p - ix - i(\beta - 1)\, x^{-1}\right). \tag{22}$$

If Ψ_E is an eigenvector of $H(\beta)$ belonging to the eigenvalue E we have seen that (vide eq. (13))

$$H(\beta + 1)\, A(\beta)\Psi_E = (E - \hbar\omega)\, A(\beta)\, \Psi_E\,, \tag{23a}$$

Similarly it can be easily shown that

$$H(\beta + 1)\, B(\beta)\Psi_E = (E + \hbar\omega)\, B(\beta)\, \Psi_E\,, \tag{23b}$$

$$H(\beta - 1)\, C(\beta)\Psi_E = (E + \hbar\omega)\, C(\beta)\, \Psi_E\,, \tag{23c}$$

$$H(\beta - 1)\, D(\beta)\Psi_E = (E - \hbar\omega)\, D(\beta)\, \Psi_E\,, \tag{23d}$$

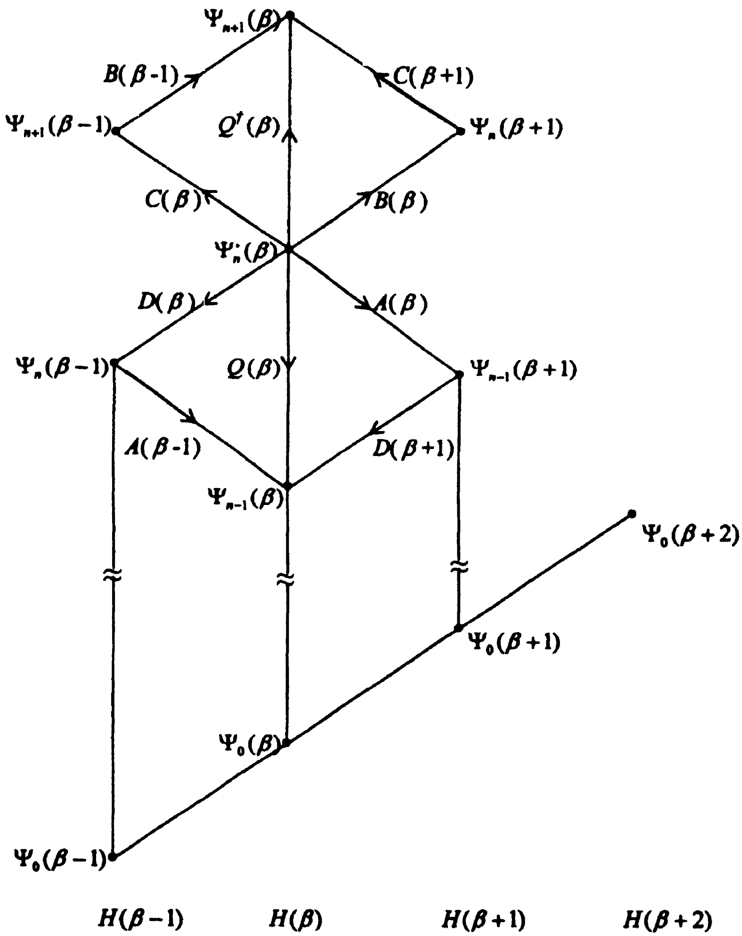

Figure 1. Laddering actions of Dirac ladder operators Q and $Q^{\dagger}$ vs laddering actions of the Infeld-Hull ladder operators A, B, C and D.

That is, $A(\beta)$, $B(\beta)$, $C(\beta)$ and $D(\beta)$ are Infeld-Hull ladder operators which take an eigenvector of $H(\beta)$ to an eigenvector of $H(\beta+1)$ or $H(\beta-1)$ with eigenvalues increased or decreased by $\hbar\omega$. These Infeld-Hull ladder operators are linear in the momentum p.

The actions of these four Infeld-Hull ladder operators and the Dirac ladder operators Q and $Q^\dagger$ are illustrated in Figure 1. It will be seen that while the Dirac ladder operators are 'vertical' (enabling to go up and down the eigenvectors of $H(\beta)$) the Infeld-Hull ladders are 'inclined' ones taking an eigenvector of $H(\beta)$ to an eigenvector of $H(\beta \pm 1)$. Figure 1 also illustrates that the Dirac ladders Q and $Q^\dagger$ can also be constructed as

$$Q^\dagger = C(\beta+1)\, B(\beta) = A^\dagger(\beta)\, B(\beta), \tag{24a}$$

$$Q = D(\beta+1)\, A(\beta) = B^\dagger(\beta)\, A(\beta), \tag{24b}$$

and also as

$$Q^\dagger = B(\beta-1)\, C(\beta) = D^\dagger(\beta)\, C(\beta), \tag{24c}$$

$$Q = A(\beta-1)\, D(\beta) = C^\dagger(\beta)\, D(\beta). \tag{24d}$$

5. Conclusion

In Section 2 we have given the Dirac ladder operators which are quadratic in momentum operator can be constructed only for a potential of the form $V(x) = Ax^2 + Bx^{-2}$ and we determine the energy eigenvalue and normalize eigenfunctions . In Section 3 we have worked out the Infeld and Hull's factorization method to determine the energy eigenvalues for this particular system of modified oscillator with a view to compare it with the Dirac laddering method. The comparison is done in Section 4 to point out the contrasting features as well as the relationship underlying the two types of ladder. The laddering process in the Dirac approach is among the eigenvectors of the particular Hamiltonian whereas in the Infeld – Hull method the laddering process connects the eigenvectors of a hierarchy of Hamiltonians ..... $H(\beta-1)$, $H(\beta)$, $H(\beta+1)$ .....

As we conclude we observe that for the eigenvalue problem for the modified oscillator we have considered here the Dirac approach is simpler and more direct than the factorization method of Infeld and Hull. However the Dirac approach, though elegant, lacks versatility, – besides the harmonic oscillator potential and the modified oscillator potential we have considered here, we do not know of any other potential which would directly admit the Dirac ladder operators.

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Appendix A: Existence of unphysical spectra

As stated in Section 4 the Hamiltonian

$$H(\beta) = \hbar\omega \frac{1}{2}\left(p^2 + x^2 + \beta(\beta-1)\, x^{-2}\right)$$

can be factorized in four different ways :

$$H(\beta) = \hbar\omega\left(A^{\dagger}A + \beta + \frac{1}{2}\right), \tag{A 1}$$

$$H(\beta) = \hbar\omega\left(B^{\dagger}B - \beta - \frac{1}{2}\right), \tag{A.2}$$

$$H(\beta) = \hbar\omega\left(C^{\dagger}C + \beta - \frac{3}{2}\right), \tag{A.3}$$

and

$$H(\beta) = \hbar\omega\left(D^{\dagger}D - \beta + \frac{3}{2}\right), \tag{A.4}$$

where, the operators A, B, C, and D are defined in eqs. (21) and (22) in the text. The first of these factorizations has been employed in the Dirac approach of Section 2 as well as in the Infeld-Hull approach of Section 3 and it leads to the physical spectrum $E_n = \hbar\omega\left(2n+\beta+1/2\right)$ (with our choice of $\beta > 1$).

In both the Dirac approach and the Infeld-Hull method the crucial input is the normalizability of the eigenvectors and the preservation of the normalizability under the action of the various operators introduced. If we leave aside this condition of normalizability just for the sake of consideration, the above four factorizations of $H(\beta)$ reveal the existence of spectra enumerated in Table 1.

Table 1. Physical and unphysical spectra

Factorisation	Eqn for $\Psi_0(x)$	$\Psi_0(x)$	$\Psi_n(x)$	Eigenvalue	Normalisability*
Eqn (A 1)	$A\Psi_0(x)=0$	$x^{\beta}e^{-\frac{1}{2}x^2}$	$\left(Q^{\dagger}\right)^n\Psi_0(x)$	$\hbar\omega\left(2n+\beta+\frac{1}{2}\right)$	Normalizable if >1
Eqn (A.2)	$B\Psi_0(x)=0$	$x^{\beta}e^{\frac{1}{2}x^2}$	$(Q)^n\Psi_0(x)$	$-\hbar\omega\left(2n+\beta+\frac{1}{2}\right)$	Not normalizable
Eqn (A.3)	$C\Psi_0(x)=0$	$x^{1-\beta}e^{\frac{1}{2}x^2}$	$(Q)^n\Psi_0(x)$	$-\hbar\omega\left(2n+\beta+\frac{1}{2}\right)$	Not normalisable
Eqn(A.4)	$D\Psi_0(x)=0$	$x^{1-\beta}e^{-\frac{1}{2}x^2}$	$\left(Q^{\dagger}\right)^n\Psi_0(x)$	$\hbar\omega\left(2n+\beta+\frac{1}{2}\right)$	Normalisable if < 0

*Normalizability is the condition for physical realizability

For a positive value of $K \equiv \beta(\beta-1)$ we may choose either $\beta > 1$ or $\beta < 0$.

$$\beta > 1 \text{ means} \qquad \beta = \beta_1 \equiv \frac{1}{2}\left(1+\sqrt{1+4K}\right), \text{ and}$$

$$\beta < 0 \text{ means} \qquad \beta = \beta_2 \equiv \frac{1}{2}\left(1-\sqrt{1+4K}\right).$$

In the text throughout we have chosen $\beta > 1$ and therefore obtained the physical spectrum in the top row in Table 1, while the remaining three spectra cannot be physically realized. Had we chosen $\beta < 0$ the physically realizable spectrum would be the one given in the bottom row of Table 1, while the other three would remain unrealizable. Of course these two physical spectra (in Table 1, top row for $\beta = \beta_1 > 1$ and the bottom row for $\beta = \beta_2 < 0$) and their eigenfunctions are exactly identical because $\beta_1 + \beta_2 = 1$. Similarly the unphysical spectra in the second row for $\beta = \beta_1$ is also identical with the unphysical spectra in the third row for $\beta = \beta_2$. The existence of these unphysical spectra with no lower bound can be traced to the fact that the operators $\Gamma_1 = 1/4\left(Q + Q^\dagger\right)$, $\Gamma_2 = i/4\left(Q - Q^\dagger\right)$ and $\Gamma_3 = G/2\hbar\omega$ form a second order realization[12] of the su (1 1) Lie algebra.